On the "finitary" Ramsey's theorem

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Abstract

We examine a version of Ramsey's theorem based on Tao, Gaspar and Kohlenbach's "finitary" infinite pigeonhole principle. We will show that the "finitary" infinite Ramsey's theorem naturally gives rise to statements at the level of the infinite Ramsey's theorem, Friedman's infinite adjacent Ramsey theorem (well-foundedness of certain ordinals up to ε_0), 1-consistency of theories up to PA and the finite Ramsey's theorem.

1 Introduction

This research is inspired by Andreas Weiermann's phase transition programme. The theme of that programme is the following curious phenomenon in first order logic:

Given a statement φ independent of some theory T, we can insert a parameter $f\colon \mathbb{N} \to \mathbb{N}$ in the statement to obtain φ_f which may be provable in, or independent of T, depending on the parameter value. When one classifies the parameter values f according to the provability of φ_f it turns out that, at a threshold value, small changes to f turns φ_f from provable in (a weak subtheory of) T to independent of T.

More information on this programme can be found at [10]. Our goal in this note is to explore the following question: *What about phase transitions for second order logic?*

A lazy answer to this question is provided by conservation results, for example: ACA_0 is conservative over PA, so any phase transition result for

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PA is also valid for ACA_0. However, we may search for more interesting cases in reverse mathematics. Reverse mathematics is the programme, started by Harvey Friedman and, among others, developed by Stephen Simpson, which aims to classify mathematics theorems according to the axioms which are required to prove them. For an introduction to reverse mathematics see [8]. In reverse mathematics we examine *equivalences.*

Again we may answer our question lazily by restating existing phase transition results, due to the fact that the independent statements used for phase transitions are known to be equivalent to the 1-consistency of the theory T under consideration. Somewhat less easily, we can also convert existing proofs of these equivalences to show the following: take $\psi_G \equiv \forall f \in G \varphi_f$ and α equal to the proof theoretic ordinal of T.

1. If $G = \{f \colon \mathbb{N} \to \mathbb{N}\}$ then $\psi(G)$ is equivalent to the well-foundedness of α.
2. If $G = \{f : f \leq \mathrm{id}\}$ then, as stated earlier, $\psi(G)$ is equivalent to the 1-consistency of the theory T.
3. If $G = \{\text{constant functions}\}$ then $\psi(G)$ is provable in RCA_0.

In this note we will examine a more interesting case, where ψ_G has parameter values for which ψ_G is independent of the well-foundedness of β for all primitive recursive ordinals β.

The starting point is Tao's "finitary" pigeonhole principle [9], which has been extensively studied in [2] from the viewpoint of reverse mathematics. We will examine a "finitary" version of Ramsey's theorem which is a generalisation of Tao's pigeonhole principle.

Definition 1 (AS**)** *A function $F \colon$ {(codes of) finite subsets of $\mathbb{N}$} $\to \mathbb{N}$ is asymptotically stable if for every sequence $X_0 \subseteq X_1 \subseteq X_2 \dots$ of finite sets, there exists i such that $F(X_j) = F(X_i)$ for all $j \geq i$.*

This definition of AS is modified from [9]. Roughly speaking, $|X| \geq F(X)$ can be interpreted as 'the finite set X is large'. AS would then be the set of possible manners in which to define 'large'.

Definition 2 (FRT_d^k**)**
For every $F \in \mathrm{AS}$ there exists R such that for all $C \colon [0, R]^d \to k$ there exists C-homogeneous H of size $> F(H)$.

Definition 3 FRT_d *is the statement* $\forall k.\mathrm{FRT}_d^k$*.* FRT *is the statement* $\forall d, k.\mathrm{FRT}_d^k$*.*

Definition 4 (RT_d^k**)**
For every $C \colon [\mathbb{N}]^d \to k$ there exists an infinite C-homogeneous set.

One can view FRT as the collection of all finite versions of RT, similar to the familiar finite Ramsey's theorem. We will show that, as is shown for the case $d = 1$ in[2], FRT^k_d is quivalent to RT^k_d over WKL_0.

Notice the following:

If, in FRT, we replace AS with the set of constant functions:

Definition 5 (CF**)**

$$\exists m.F = m,$$

the resulting theorem becomes simply the finite Ramsey's theorem.

If we replace AS with the following:

Definition 6 (UI**)**

$$\exists m \forall X.F(X) \leq \max\{\min X, m\},$$

then the resulting theorem is the Paris–Harrington principle, which, for dimension $d + 1$ is equivalent to the 1-consistency of $\mathrm{I}\Sigma_d$. It is equivalent to 1-consistency of PA for unrestricted dimensions.

Definition 7 $\mathrm{FRT}^k_d(G)$ *is the statement obtained from* FRT^k_d *by replacing* $F \in \mathrm{AS}$ *with* $F \in G$. $\mathrm{FRT}_d(G)$ *is the statement* $\forall k.\mathrm{FRT}^k_d(G)$. $\mathrm{FRT}(G)$ *is the statement* $\forall d, k.\mathrm{FRT}^k_d(G)$.

One obvious question is whether there are properties G such that the strength of $\mathrm{FRT}(G)$ lies strictly between $\mathrm{FRT}(\mathrm{UI})$ and $\mathrm{FRT}(\mathrm{AS})$. We will show that this is the case for:

Definition 8 (MD**)**

$$\forall X, Y. \min X = \min Y \to F(X) = F(Y).$$

Because this latter version has connections with Friedman's adjacent Ramsey theorem we conclude with determining the level-by-level strength of the adjacent Ramsey theorem.

2 FRT

We assume familiarity with reverse mathematics, primitive recursion, RCA_0, WKL_0 and Ramsey's theorem as in Chapters II and IV in [8]. Please note

that for finite set X we also use X to denote its code.

The main theorem in this section is:

Theorem 9
(a) $\mathrm{RCA}_0 \vdash \mathrm{FRT}^k_d \to \mathrm{RT}^k_d$,
(b) $\mathrm{WKL}_0 \vdash \mathrm{RT}^k_d \to \mathrm{FRT}^k_d$.

We will make use of:

Lemma 10 *The following are primitive recursive:*
1. *the relation* $\{(x, X) : x \in X\}$,
2. *the relation* $\{(X, Y) : X \subseteq Y\}$,
3. *the relation* $\{(X, C) : X$ *is* C*-homogeneous*$\}$ *and*
4. *the function* $(x, C) \mapsto C(x)$ *for finite functions* C.

Proof: Exercise for the reader.

□

Proof of Theorem 9 (a): We adapt the proof of the case $d = 1$ from [2]. Please notice the extra steps needed to deal with the modified definition of AS.

In RCA_0, we show $\neg\mathrm{RT}^k_d \to \neg\mathrm{FRT}^k_d$. Suppose $C\colon [\mathbb{N}]^d \to k$ is a colouring such that every C-homogeneous set has finite size. Define the following F primitive recursively:

$$F(X) = \begin{cases} |X| + 1 & \text{if } X \text{ is } C\text{-homogeneous,} \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1: $F \in \mathrm{AS}$. Take $X_0 \subseteq X_1 \subseteq \ldots$. Examine the Σ^0_1 formula:

$$\varphi(n) \equiv \exists i(n \in X_i).$$

By Lemma II.3.7 of [8] $\{n : \varphi(n)\}$ is finite or there exists a one-to-one function f such that

$$\forall n[\varphi(n) \leftrightarrow \exists m(f(m) = n)].$$

If $\{n : \varphi(n)\}$ is finite then there exists i with $F(X_i) = F(X)$ and we are finished with the claim, so assume the latter case.

We will show that there exists an infinite set X such that $n \in X \to \varphi(n)$ (hence X is a subset of the possibly nonexistent $\bigcup X_i$). This is sufficient, because then $\forall i \exists j > i F(X_j) \neq F(X_i)$ implies X is C-homogeneous.

We show this by translating a rather common exercise from computability theory to our context: Given an infinite recursively enumerable set, show that it contains an infinite decidable subset.

Take Σ_1^0 formula:

$$\phi(n) \equiv \exists m[f(m) \geq n \wedge f(\mu x \leq m.f(x) \geq n) = n].$$

and Π_1^0 formula:

$$\psi(n) \equiv \forall m[f(m) \geq n \to f(\mu x \leq m.f(x) \geq n) = n].$$

These two formulas are equivalent by unboundedness of f, so by Δ_1^0-comprehension the infinite set $X = \{n : \phi(n)\}$ exists. This finishes the proof of claim 1.

Claim 2: F is a counterexample for FRT_d^k. Take arbitrary R, Define $D = C$ restricted to $[0, R]^d$. By definition of F any D-homogeneous set H has size $< F(H)$, ending the proof of claim 2 and part (a) of the theorem.

Proof of Theorem 9 (b): We use a compactness proof which involves König's lemma. However, we take care that the application of König's lemma uses only the bounded version (hence we reason in WKL_0 by Lemma IV.1.4 in [8]).

Assume $\neg\mathrm{FRT}_d^k$, hence there exists $F \in \mathrm{AS}$ such that for all R there exists $C\colon [0, R]^d \to k$ for which every C-homogeneous set $H \subseteq [0, R]$ has size $\leq F(H)$. Enumerate such colourings with $\{C_{R,i}\}_{i \leq n_R}$. Notice that the codes of these colourings can be bounded by some function which is primitive recursive in d, k, R. We define the following bounded (by previous remark) and infinite tree:

$$T = \{\langle C_{1,i_1}, \ldots, C_{R,i_R}\rangle : C_{1,i_1} \subseteq \cdots \subseteq C_{R,i_R}\}.$$

Take the colourings $D_1 \subseteq D_2 \subseteq \ldots$ from the infinite path in T, which exists due to bounded König's lemma. Define $D\colon [\mathbb{N}]^d \to k$ as follows:

$$D(x) = D_{\max x}(x).$$

Claim: D is a counterexample for RT_d^k. Assume H is D-homogeneous. By construction of T and $D = \bigcup D_i$, the size of $H_i = H \cap [0, i]$ is less than or equal to $F(H_i)$ for every i. Note that $H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$ and $H = \bigcup H_i$, so (by $F \in \mathrm{AS}$) there exists i such that $F(H_j) = F(H_i)$ for all $j \geq i$, hence H is finite. This ends the proof of the claim and part (b) of the theorem.

□

Question 11 *Is* WKL_0 *required in part (b) of this theorem? Notice that* WKL_0 *is not required for* $d \geq 3$*.*

3 Restriction to the minimally dependent

We assume basic familiarity with ordinals up to ε_0 and their cantor normal forms.

Definition 12 $\omega_0 = 1$ *and* $\omega_{n+1} = \omega^{\omega_n}$*.*

Definition 13 ($\mathrm{WO}(\alpha)$**)** *Every infinite sequence* $\alpha_0, \alpha_1, \ldots$ *below* α *has* $i < j$ *such that* $\alpha_i \leq \alpha_j$*.*

The main theorem in this section is:

Theorem 14 $\mathrm{RCA}_0 \vdash \mathrm{WO}(\omega_d) \leftrightarrow \mathrm{FRT}_d(\mathrm{MD})$

Observe first that $\mathrm{FRT}_d(\mathrm{MD})$ is equivalent to $\forall f\colon \mathbb{N} \to \mathbb{N}.\mathrm{PH}^d_f$.

Definition 15 (PH^d_f**)** *For all* a *there exists* R *such that for all* $C\colon [a, R]^d \to k$ *there exists* C*-homogeneous* H *of size* $f(\min H)$*.*

Proof of Theorem 14: '$\rightarrow$' in Subsection 3.2
'$\leftarrow$' in Subsection 3.1.

□

3.1 Lower bound

We modify the proof of $\mathrm{PH}_{\mathrm{id}} \to \mathrm{Tot}(H_{\varepsilon_0})$ from [1]. The proof below consist mostly of recalling the necessary definitions and lemmas, where the final step is modified to fit our new situation. We skip the proofs when they are unchanged from the original.

Definition 16 *Given* $\alpha = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_n} \cdot a_n$ *and* $\beta = \omega^{\beta_1} \cdot b_1 + \cdots + \omega^{\beta_m} \cdot b_m$*, with the* a_i, b_i *positive integers,* $\alpha_1 > \cdots > \alpha_n$ *and* $\beta_1 > \cdots > \beta_m$ *we define:*

1. *The comparison position* $\mathrm{CP}(\alpha, \beta)$ *is the smallest* i *such that* $\omega^{\alpha_i} \cdot a_i \neq \omega^{\beta_i} \cdot b_i$ *if such an* i *exists, zero otherwise.*
2. *The comparison coefficient* $\mathrm{CC}(\alpha, \beta)$ *is* $a_{\mathrm{CP}(\alpha,\beta)}$*, where* $a_0 = 0$*.*
3. *The comparison exponent* $\mathrm{CE}(\alpha, \beta)$ *is* $\alpha_{\mathrm{CP}(\alpha,\beta)}$*, where* $\alpha_0 = 0$*.*

Define the maximal position MP *and maximal coefficient* MC *by induction on* α *as follows:* $\mathrm{MP}(0) = 1$ *and* $\mathrm{MC}(0) = 0$. *Given* $\alpha = \omega^{\alpha_1} \cdot a_1 + \cdots + \omega^{\alpha_n} \cdot a_n > 0$, *with the* a_i *positive integers and* $\alpha_1 > \cdots > \alpha_n$, *define:*

(4) $\mathrm{MP}(\alpha) = \max\{n, \mathrm{MP}(\alpha_i)\}$.

(5) $\mathrm{MC}(\alpha) = \max\{a_i, \mathrm{MC}(\alpha_i)\}$.

Lemma 17 *We have:*

1. $\mathrm{CP}(\alpha, \beta) \leq \mathrm{MP}(\alpha)$.
2. $\mathrm{CC}(\alpha, \beta) \leq \mathrm{MC}(\alpha)$.
3. $\mathrm{MP}(\alpha_i) \leq \mathrm{MP}(\alpha)$ *and* $\mathrm{MC}(\alpha_i) \leq \mathrm{MC}(\alpha)$.
4. $\mathrm{CP}(\alpha, \beta) \leq \mathrm{CP}(\beta, \gamma) \wedge \mathrm{CE}(\alpha, \beta) \leq \mathrm{CE}(\beta, \gamma) \wedge \mathrm{CC}(\alpha, \beta) \leq \mathrm{CC}(\beta, \gamma) \Rightarrow \alpha \leq \beta$.

Definition 18 *Let* l, d, n *be nonnegative integers. Define* $\omega_0(l) = l$ *and* $\omega_{n+1}(l) = \omega^{\omega_n(l)}$. *Define* $F^l_d \colon \omega_d(l+1)^d \to \mathbb{N}^{2d+l-1}$ *by recursion on* d*:*

1. *Given* $\alpha = \omega^l \cdot n_l + \cdots + \omega^0 \cdot n_0$, *define* $F^l_1(\alpha) = (n_l, \dots, n_0)$.
2. $F^l_{d+1}(\alpha_1, \dots, \alpha_{d+1}) =$
   $(\mathrm{CP}(\alpha_1, \alpha_2), \mathrm{CC}(\alpha_1, \alpha_2), F^l_d(\mathrm{CE}(\alpha_1, \alpha_2), \dots, \mathrm{CE}(\alpha_d, \alpha_{d+1})))$.

Lemma 19 $F^l_d(\alpha_1, \dots, \alpha_d) \leq F^l_d(\alpha_2, \dots, \alpha_{d+1}) \Rightarrow \alpha_1 \leq \alpha_2$.

Lemma 20 $F^l_d(\alpha_1, \dots, \alpha_d) \leq \max\{\mathrm{MC}(\alpha_1), \mathrm{MP}(\alpha_1)\}$.

We are finally ready to finish the proof the lower bound of Theorem 14. The following lemma is where the proof from [1] is modified:

Lemma 21 $\mathrm{RCA}_0 \vdash \forall f. \mathrm{PH}^d_f \to \mathrm{WO}(\omega_d)$, *where* PH^d_f *is* PH_f *with fixed dimension* d.

Given infinite sequence $\alpha_0, \alpha_1, \alpha_2, \dots$ below $\omega_d(l+1)$ take

$$f(i) = \max\{\mathrm{CC}(\alpha_i), \mathrm{CP}(\alpha_i)\} + d + 2$$

and R from PH_f in dimension $d+1$, $a = 0$ and $c = 2d + l$. Define colouring $C \colon [0, R]^{d+1} \to [0, 2d + l]$:

$$C(x_1, \dots, x_{d+1}) = \begin{cases} 0 & \text{if } F^l_d(\alpha_{x_1}, \dots, \alpha_{x_d}) \leq \\ & \quad F^l_d(\alpha_{x_2}, \dots, \alpha_{x_{d+1}}), \\ i & \text{otherwise,} \end{cases}$$

where i is the least such that:

$$(F^l_d(\alpha_{x_1}, \dots, \alpha_{x_d}))_i > (F^l_d(\alpha_{x_2}, \dots, \alpha_{x_{d+1}}))_i.$$

Observe that $(F^l_d(\alpha_{x_1},\dots,\alpha_{x_d}))_i \leq \max\{\mathrm{CC}(\alpha_{x_1}),\mathrm{CP}(\alpha_{x_1})\}$ (this is a consequence of Lemma 20). Take homogeneous H of size $f(\min H)$from PH_f. If the value of C on $[H]^{d+1}$ is $i > 0$ we can obtain a decending sequence of natural numbers below $f(\min H)-d-2$ of length $f(\min H)-d$, which is impossible. Hence the value of C is 0, delivering $\alpha_{x_1} \leq \alpha_{x_2}$.

□

3.2 Upper bound

We use the upper bounds result from Section 6 in [3], observing that, mostly thanks to the formalisation of large parts in $\mathrm{I}\Sigma_1$ in Section II.3 in [7], the proofs are within $\mathrm{RCA}_0 + \mathrm{WO}(\omega_d)$. Alternatively, one can use Corollary 15 from [6], which states that the theorem in question is provable in RCA_0.

A similar version, called relativised Paris–Harrington for $d = 2$ has also been studied by Kreuzer and Yokoyama in [4].

Definition 22 $A = \{a_0 < \cdots < a_b\}$ *is* α*-large if* $\alpha[a_0]\dots[a_b] = 0$*, where* $\alpha[.]$ *denotes the canonical fundamental sequences for ordinals below* ε_0*.*

Lemma 23 RCA_0 *proves the following: if* $\mathrm{WO}(\omega_d)$ *then for every strictly increasing* $f\colon \mathbb{N} \to \mathbb{N}$*,* $a \in \mathbb{N}$*,* $\alpha < \omega_d$ *there exists* α*-large set* $\{f(a), f(a+1),\dots,f(b)\}$*.*

Proof: Define the following descending sequence of ordinals: $\alpha_0 = \alpha$ and:

$$\alpha_{i+1} = \alpha_i[f(i)].$$

By well-foundedness of ω_d this sequence reaches zero, delivering the desired α-large set.

□

Assume without loss of generality, that f is strictly increasing and > 3. By $\mathrm{WO}(\omega_d)$ there exists $\omega_{d-1}(c+5)$-large set $A = \{f(a),\dots,f(b)\}$. We claim that $R = b$ witnesses PH^d_f: Take colouring $C\colon [a,R]^d \to c$, define $D\colon [A]^d \to c$ as follows:

$$D(x_1,\dots,x_d) = C(f^{-1}(x_1),\dots,f^{-1}(x_d)).$$

By Theorem 6.7 from [3] or Corollary 15 from [6] there exists D-homogeneous X with size $\min X$. Then $H = \{f^{-1}(x) : x \in X\}$ is C-homogeneous and of size $f(\min H)$. This ends the proof of Theorem 14.

□

4 FRT and adjacent Ramsey

Definition 24 *For r-tuples $a \le b$ denotes the coordinatewise ordering:*

$$a \le b \Leftrightarrow (a)_1 \le (b)_1 \wedge \cdots \wedge (a)_r \le (b)_r.$$

Definition 25 (AR_d) *For every $C\colon \mathbb{N}^d \to \mathbb{N}^r$ there exist $x_1 < \cdots < x_{d+1}$ such that $C(x_1, \ldots, x_d) \le C(x_2, \ldots x_{d+1})$.*

Definition 26 AR *denotes* $\forall d.\mathrm{AR}_d$.

In this section we will show that:

Theorem 27 $\mathrm{RCA}_0 \vdash \mathrm{WO}(\omega_{d+1}) \leftrightarrow \mathrm{AR}_d$

Proof: '$\leftarrow$': We use F_d^l from 3.1. Given sequence of ordinals $\omega_d(l+1) > \alpha_0, \alpha_1, \ldots$ define:

$$C(x_1, \ldots, x_d) = F_d^l(\alpha_{x_1}, \ldots, \alpha_{x_d}).$$

By AR_d there exist $x_1 < \cdots < x_{d+1}$ with $C(x_1, \ldots, x_d) \le C(x_2, \ldots, x_{d+1})$, which by Lemma 19 deliver $\alpha_{x_1} \le \alpha_{x_2}$.

'$\rightarrow$': By Theorem 14 it is sufficient to show that $\forall f.\mathrm{PH}_f^{d+1} \to \mathrm{AR}_d$. For this it is sufficient to simply note that the proof of $\mathrm{PH}_{d+1} \to \mathrm{AR}_d$ from [1] (please note the difference in AR as defined there) works fine when relative to the function

$$f(x) = \max_{y \in \{0, \ldots, x\}^d} C(y).$$

Replace the strong adjacent Paris–Harrington principle with a version relative to f:

Definition 28 (SAPH_f^d) *For every $c, k.m$ there exists an R such that for every colouring $C\colon [m, \ldots, R]^d \to [0, c]$ there exists C-homogeneous $H = \{h_1 < h_2 < \ldots\}$ of size $f(h_k)$.*

Then $\forall f.\mathrm{PH}_f^{d+1} \to \forall f.\mathrm{SAPH}_f^{d+1} \to \mathrm{AR}_d$ by copying the proofs of Theorems 3.4 and 3.5 from [1].

$\square$

5 Conclusions

RCA_0 proves the following:

FRT	$\leftrightarrow$	RT		
FRT^k_d	$\leftarrow$	RT^k_d for (d>2)		
FRT^k_d	$\rightarrow$	RT^k_d		
$\mathrm{FRT(MD)}$	$\leftrightarrow$	AR	$\leftrightarrow$	$\mathrm{WO}(\varepsilon_0)$
$\mathrm{FRT}_{d+1}(\mathrm{MD})$	$\leftrightarrow$	AR_d	$\leftrightarrow$	$\mathrm{WO}(\omega_{d+1})$
$\mathrm{FRT(UI)}$	$\leftrightarrow$	1-consistency of PA		
$\mathrm{FRT}_{d+1}(\mathrm{UI})$	$\leftrightarrow$	1-consistency of $\mathrm{I}\Sigma_d$		
$\mathrm{FRT(CF)}$				

The last three of those lines are true because $\mathrm{FRT}_d(\mathrm{UI})$ is equivalent to $\mathrm{PH}^d_{\mathrm{id}}$, so the equivalence to 1-consistency is the classic Paris–Harrington result from [5].

Furthermore, $\mathrm{WKL}_0 \vdash \mathrm{RT}^k_d \rightarrow \mathrm{FRT}^k_d$.

Corollary 29 *Over* RCA_0*:*

$$\mathrm{FRT(CF)} < \mathrm{FRT(UI)} < \mathrm{FRT(MD)} < \mathrm{FRT(AS)}.$$

Question 30 *Do the same implications hold for* RCA^*_0 *and, where* WKL_0 *is used, in* WKL^*_0*?*

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